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AN APPROXIMATE METHOD FOR ANALYSING STOCHASTIC MECHANICAL SYSTEMS†

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An approximate method for analysing diffusion processes in a natural mechanical system when there are perturbing forces similar to normal white noise is proposed. It is based on orthogonal expansions of the one-dimensional probability density of the state vector in a suitable Hilbert space of functions which are square-integrable with respect to a certain measure in the phase space (manifold) of the system. The method consists of solving a special system of linear ordinary differential equations for the expansion coefficients, and is suitable for computer implementation. The method is rigorously proved. The motion of a two-dimensional mathematical pendulum in a random medium is investigated as an example.

1. FORMULATION OF THE PROBLEM

CONSIDER a natural mechanical system subject to time-independent geometrical constraints. Let Q be a (smooth) configurational manifold of positions, of dimension n, and $\mathbf{q} = [q_1, \ldots, q_n]^t$ are local coordinates on Q. We will assume that the system is subject to the action of conservative forces with potential energy $\Pi(\mathbf{q})$, dissipative forces which are derivatives of the Rayleigh function $\mathbf{q}^{\cdot t}\mathbf{D}(\mathbf{q})\mathbf{q}^{\cdot}/2$, and also random forces, represented by a random vector $\mathbf{F} = \mathbf{b}(\mathbf{q})\mathbf{V}(t)$, where $\mathbf{V}(t) = [V_1 \ldots V_l]^t$ is a vector of normally distributed white noise with constant intensity matrix ν (of dimensions $l \times l$), $\mathbf{b}(\mathbf{q})$ a certain matrix-valued function of dimensions $n \times l$.

We will write the equations of motion of the system in Hamiltonian form

$$\mathbf{q} = \partial H/\partial \mathbf{p}, \quad \mathbf{p}^{\cdot} = -\partial H/\partial \mathbf{q} - \mathbf{D}^*(\mathbf{q})\mathbf{p} + \mathbf{b}(\mathbf{q})\mathbf{V}(t), \quad \mathbf{q}(t_0) = \mathbf{q}_0, \quad \mathbf{p}(t_0) = \mathbf{p}_0 \tag{1.1}$$
$$H = \mathbf{p}^t\Omega(\mathbf{q})\mathbf{p}/2 + \Pi(\mathbf{q}), \quad \mathbf{D}^* = \mathbf{D}\Omega$$

where H is the Hamiltonian and Ω is some positive matrix.

We will further assume that all the deterministic functions $\mathbf{D}(\mathbf{q})$, $\Omega(\mathbf{q})$, $\mathbf{b}(\mathbf{q})$, $\Pi(\mathbf{q})$ are sufficiently smooth.

Equations (1.1) are Ito stochastic differential equations in the manifold $X = T^*Q$ (the phase space of the system) [1, 2]. They define a diffusion process controlled by a second-order operator which is parabolic in the wide sense

$$G = \frac{\partial^t H}{\partial \mathbf{p}}\frac{\partial}{\partial \mathbf{q}} - \left(\frac{\partial^t H}{\partial \mathbf{q}} + \mathbf{p}^t\mathbf{D}^*\right)\frac{\partial}{\partial \mathbf{p}} + \frac{1}{2}\operatorname{tr}\left(\sigma\frac{\partial}{\partial \mathbf{p}}\frac{\partial^t}{\partial \mathbf{p}}\right), \quad \sigma = \mathbf{b}\nu\mathbf{b}^t$$

The one-dimensional probability density $f(\mathbf{x}, t)$ of the process $\mathbf{x}(t) = [\mathbf{q}(t)^t\mathbf{p}(t)^t]^t$ relative to a phase volume element $d\mu_0 = dq_1 \ldots dq_n dp_1 \ldots dp_n$ satisfies the Fokker–Planck–Kolmogorov equation $\partial f/\partial t = G^*F$, where G^* is the adjoint of the infinitesimal operator G of the process $\mathbf{x}(t)$. Written in full, this equation is

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$$\frac{\partial f}{\partial t} + \frac{\partial^t}{\partial \mathbf{q}}\left(\frac{\partial H}{\partial \mathbf{p}} f\right) - \frac{\partial^t}{\partial \mathbf{p}}\left[\left(\frac{\partial H}{\partial \mathbf{q}} + \mathbf{D}^* \mathbf{p}\right) f\right] - \frac{1}{2}\,\mathrm{tr}\left(\sigma \frac{\partial}{\partial \mathbf{p}} \frac{\partial^t}{\partial \mathbf{p}} f\right) = 0 \tag{1.2}$$

$$\int_X f(\mathbf{x}, t) d\mu_0 = 1, \quad f(\mathbf{x}, t_0) = f_0(\mathbf{x})$$

It is not possible to solve Eq. (1.2) exactly in all generality. We therefore set ourselves the task of finding an approximation to the one-dimensional distribution of the state vector.

There is a considerable amount of literature on approximate methods for analysing stochastic differential systems (SDS) (for a survey, see, for example [3]). Most of these methods, however, are based on biorthogonal expansions of the density f with a weight function that depends on unknown parameters (the moments of the process), so that they are difficult to justify in the rigorous sense and to extend to diffusion in manifolds. In addition, authors who have considered orthogonal expansions of f have generally confined their attention to the case $X = R^m$ and assumed that the right-hand sides of the equations are polynomials. A common weakness of these methods is the lack of a rigorous justification.

Another efficient approach to the analysis of SDS is the method of averaging (see [4] for its application to some problems of mechanics). Here, however, one must limit oneself to situations in which the slow motion is one-dimensional.

Some qualitative conclusions as to the behaviour of the system may be derived from the following theorem of Khas'minskii [5]: if a compact set $K \subset X$ and a function $U(\mathbf{x}) \geq 0$ exist such that $GU(\mathbf{x}) \leq -1$ for $\mathbf{x} \in K$, then a steady-state solution exists. This solution is unique and ergodic, i.e. it is the (weak) limit of any initial distribution (see also [6]). For example, if Q is compact and the dissipation is complete, the assumptions of Khas'minskii's theorem are satisfied. The role of $U(\mathbf{x})$ may be assigned to the Hamiltonian H.

The method proposed below is based on orthogonal expansions of the density f and enables one, by solving a certain system of linear ordinary differential equations of order N, to find approximate expressions for the one-dimensional density f and for some moment characteristics of $\mathbf{x}(t)$. Conditions will be indicated under which the approximation error (in the sense of the metric of a certain Hilbert space) will tend to zero as $N \to \infty$. The method is suitable for computer implementation.

2. METHOD OF INVESTIGATION

Let $L_2(X, \mu)$ be the standard Hilbert space of functions $\xi(\mathbf{x})$ on X with measure $\mu (d\mu = \mu_1(\mathbf{x}) d\mu_0)$ and norm $\|\xi\| = [\int_X |\xi(\mathbf{x})|^2 d\mu]^{1/2}$, with some orthonormal basis $\{e_j(\mathbf{x})\}$ [7]. The index j may be either a vector (in which case j is actually a multi-index varying in some set) or a scalar (then $j = 0, 1, 2, \ldots$). Clearly, as the basis is denumerable, vector indices may always be replaced by scalars and vice versa. In this section we shall use the scalar notation.

Let us assume regarding the unknown density that $f(\mathbf{x}, t)/\mu_1(\mathbf{x}) \in L_2(X, \mu)$. If system (1.1) is smooth, this may always be assured by a suitable shoice of the measure μ. Then a unique series $\Sigma_j c_j e_j(\mathbf{x})$ exists that converges in the metric of $L_2(X, \mu)$ (i.e. in mean square) to $f(\mathbf{x}, t)/\mu_1(\mathbf{x})$, $f/\mu_1 \sim \Sigma_j c_j e_j$, where the coefficients c_j are given by Fourier's formulae

$$c_j(t) = \int_X f(\mathbf{x}, t) e_j(\mathbf{x}) d\mu_0 = Me_j(\mathbf{x}) \tag{2.1}$$

(M is the expectation operator). The coefficients $c_j(t)$ satisfy the following denumerably infinite system of linear ordinary differential equations

$$c_j^{\cdot} = (Me_j)^{\cdot} = Me_j^{\cdot} = MGe_j \tag{2.2}$$

Let us assume that $Ge_j \in L_2(X, \mu)$; then Ge_j may also be expanded in a convergent series

$$Ge_j \sim \sum_i a_{ji} e_i \qquad \left(a_{ji} = \int_X e_i G e_j d\mu = \mathrm{const}\right)$$

and system (2.2) becomes

$$c_j^{\cdot} = \sum_i c_i a_{ji}, \quad c_j(t_0) = \int_X f_0(\mathbf{x}) e_j(\mathbf{x}) d\mu_0 \tag{2.3}$$

This system is linear but not homogeneous, since f must satisfy a normalizing condition. Most frequently, the first of the coefficients c_j is simply equal to a known constant (we shall indeed assume that this is the case here).

It is generally impossible to find an exact solution of the general equation (2.3). We can therefore confine our attention to the expansion coefficients c_j up to and including some order N (the other coefficients are assumed to vanish)

$$c_j^{*\cdot} = \sum_{i=0}^{N} c_i^* a_{ji} \qquad (j = 1, \ldots, N) \tag{2.4}$$

Solving this system with suitable initial conditions, we obtain an approximate value of the one-dimensional density

$$f^*(\mathbf{x}, t) = \mu_1(\mathbf{x}) \sum_{j=0}^{N} c_j^*(t) e_j(\mathbf{x})$$

A singular point of system (2.4) represents a steady-state solution.

3. JUSTIFICATION OF THE METHOD

To justify the method we must estimate the distance between $c_j^*(t)$ and $c_j(t)$, and between $f(\mathbf{x}, t)$ and $f^*(\mathbf{x}, t)$, as a function of the order N of the approximation.

Let us consider the first N exact equations (2.3) for c_j ($j = 1, \ldots, N$) and compare them with the approximations (2.4) (the initial conditions match). For each j, the exact equation differs from its approximation by the quantity $\alpha_j(t) = \Sigma_{i>N} c_i(t) a_{ji}$. Let us assume that the series $\Sigma_j c_j(t) e_j(\mathbf{x})$ converges to f/μ_1 (in the $L_2(X, \mu)$-norm) uniformly in t. Then, since

$$\sum_{i=0}^{\infty} c_i^2 = \left\| \frac{f}{\mu_1} \right\|^2, \qquad \sum_{i=0}^{\infty} a_{ji}^2 = \| Ge_j \|^2$$

it follows from the Cauchy–Schwarz inequality that

$$\left| \sum_{i>N} c_i(t) a_{ji} \right| \leqslant \sqrt{\sum_{i>N} c_i^2(t)} \sqrt{\sum_{i>N} a_{ji}^2} \tag{3.1}$$

and moreover

$$\sum_{i>N} c_i^2(t) \leqslant \epsilon(N), \qquad \sum_{i>N} a_{ji}^2 = \chi_j(N)$$

with $\epsilon(N) \to 0$ as $N \to \infty$.

Let $\chi(N) = \max_{1 \leqslant j \leqslant N} |\chi_j(N)|$. Then it follows from (3.1) that

$$|\alpha_j(t)| \leqslant \epsilon_1(N), j = 1, \ldots, N, \ \epsilon_1(N) \equiv \sqrt{\epsilon(N)\chi(N)}$$

Thus, the system of the first N exact equations (2.3) and the system of N approximate linear differential equations with constant coefficients differ by a certain column vector $\alpha(t) = [\alpha_1 \ldots \alpha_N]^t$, which is bounded as a function of t. Let us assume that all the real parts of the eigenvalues of the matrix $\mathbf{A}_N$ of system (2.4) are negative, i.e. the approximate system is stable with respect to persistent perturbations. Then

$$\kappa_1(N, \epsilon_1) = \sup_{B} \sup_{t \geqslant t_0} \exp(t\mathbf{A}_N) \int_{t_0}^{t} \exp(-\tau \mathbf{A}_N) \alpha(\tau) d\tau \tag{3.2}$$

exists, where the infimum is taken over the set B of smooth vector-valued functions $\alpha(t)$ that are bounded as functions of t by the positive number ϵ_1 (that is, $|\alpha_j(t)| \leqslant \epsilon_1, j = 1, \ldots, N$).

Let $\kappa(N, \epsilon_1)$ denote the maximum component of the N-vector κ_1. This quantity $\kappa(N, \epsilon_1)$ characterizes the transfer properties of the approximate system (2.4). For simplicity, we shall estimate $\kappa(N, \epsilon_1)$ on the assumption that all the eigenvalues λ_i $(i = 1, \ldots, N)$ of $\mathbf{A}_N$ are simple. In that case, R^N [the phase space of system (2.4)] splits as a direct sum of one- and two-dimensional invariant subspaces for $\mathbf{A}_N$, and there is a real transformation of variables $\mathbf{u} = \Psi\mathbf{c}$, that brings system (2.4) to canonical form (see, e.g. [8]). In the new variables $\mathbf{u}$, the matrix of the linear system has along its principal diagonal either real eigenvalues or 2 by 2 blocks

$$\begin{Vmatrix} \beta_1 & \beta_2 \\ -\beta_2 & \beta_1 \end{Vmatrix}$$

[corresponding to eigenvalues of the form $\beta_1 \pm i\beta_2$ $(\beta_1 < 0)$]. It is now quite easy to estimate κ for each subsystem. The result, obtained by combining the results for the entire system is

$$\kappa(N, \epsilon_1) \leqslant \sqrt{2}\,\epsilon_1 N^2 \kappa_2, \quad \kappa_2 = \max_{i,j} |\psi_{ij}| \max_{i,j} |\psi^{ij}| / \min_i |\operatorname{Re} \lambda_i| \tag{3.3}$$

where ψ_{ij} and ψ^{ij} $(i, j = 1, \ldots, N)$ are the elements of the matrices Ψ and Ψ^{-1}, respectively.

Note that Ψ is the matrix of the transformation to a coordinate system consisting of eigenvectors of the operator $\mathbf{A}_N$ (if they are complex, take their real and imaginary parts separately). If one requires these to be unit vectors, then $\max_{i,j}|\psi_{ij}| = 1$.

The estimation of κ when some eigenvalues are multiple is similar.

The system consisting of the first N exact equations (2.3) is obtained by perturbing the approximate system (2.4) by a certain bounded vector $\alpha(t)$. Since all the eigenvalues of $\mathbf{A}_N$ are negative, it follows from the formula for the general solution of an inhomogeneous linear equation [8] and from (3.2) that the solutions of the exact and approximate systems in this case differ by at most $\kappa(N, \epsilon_1(N))$, i.e.

$$|c_j(t) - c_j^*(t)| \leqslant \kappa(N, \epsilon_1(N)) \; (j = 1, \ldots, N)$$

Consequently, if $\kappa(N, \epsilon_1(N)) \to 0$ as $N \to \infty$, then also $c_j \to c_j^*$ $(j = 1, \ldots, N)$ uniformly in t.

We will now estimate the distance between $f(\mathbf{x}, t)$ and $f^*(\mathbf{x}, t)$. We will use the Parseval equality

$$\| (f - f^*)/\mu_1 \|^2 = \sum_{j=0}^{N} (c_j - c_j^*)^2 + \sum_{j > N} c_j^2$$

It implies that, under all our assumptions

$$\|(f - f^*)/\mu_1\|^2 \leqslant N\kappa^2(N, \epsilon_1(N)) + \epsilon(N)$$

If we require in addition that $N\kappa^2(N, \epsilon_1(N)) \to 0$ as $N \to \infty$, then also $\|(f - f^*)/\mu_1\| \to 0$ as $N \to \infty$.

We have thus proved the following theorem.

Theorem 1. Suppose that the initial SDS (1.1) satisfies the following conditions:

1. $f(\mathbf{x}, t)/\mu_1(\mathbf{x}) \in L_2(X, \mu)$ and the Fourier series of this function converges to it uniformly in t.
2. $Ge_j(\mathbf{x}) \in L_2(X, \mu)$ for all j.
3. All the real parts of the eigenvalues of the matrix $\mathbf{A}_N$ of the approximate system (2.4) are negative for any sufficiently large N.
4. $\lim_{N\to\infty} \kappa(N, \epsilon_1(N)) = 0$.

Then $\lim_{N\to\infty} |c_j(t) - c_j^*(t)| = 0$ $(j = 1, \ldots, N)$ uniformly in t $(t \geqslant t_0)$.

If we add the next condition.

5. $\lim_{N\to\infty} N\kappa^2(N, \epsilon_1(N)) = 0$ then $\lim_{N\to\infty} \|[f(\mathbf{x}, t) - f^*(\mathbf{x}, t)]/\mu_1(\mathbf{x})\| = 0$ uniformly in t.

Considering inequality (3.3), we can replace conditions 4 and 5, respectively, by

$$\lim_{N\to\infty} N^2\kappa_2(N)\sqrt{\epsilon(N)\chi(N)} = 0 \text{ and } \lim_{N\to\infty} N^5\kappa_2^2(N)\epsilon(N)\chi(N) = 0 \tag{3.4}$$

A sufficient condition for these conditions to hold is that the "contractive" properties of the operator $\mathbf{A}_N$ and the rate of decrease of the expansion coefficients of f/μ_1 as N increases have suppressed the effect of the increasing dimension of the system (which is a power of N) and, as follows from (3.3), certain effects due to the orientation in R^N of the eigenfactors of $\mathbf{A}_N$.

The condition for the Fourier series of f/μ to be uniformly convergent implies that

$$\sup_{t_0 \leqslant t < \infty} \sum_{i > N} c_i^2(t) \to 0 \qquad \text{as} \quad N \to \infty$$

This condition may be weakened. If

$$\sup_{t_0 \leqslant t < T} \sum_{i > N} c_i^2(t) \to 0 \qquad \text{as} \quad N \to \infty$$

then all our conclusions remain true for $t \in [t_0, T]$.

The theorem will also remain true if the sequence of natural numbers N is replaced by a subsequence N_j; the conclusion will then hold as $j \to \infty$.

We will now discuss some aspects of the practical application of Theorem 1. Condition 2 can be verified directly; the same holds for the estimation of $\chi(N)$. The term $\epsilon(N)$ depends on the smoothness of $f(\mathbf{x}, t)$, which in turn depends on the smoothness of the coefficients of the initial system (see Sec. 5 below). Some difficulties may be involved in (analytically) checking conditions 3–5. In practice, however, there is no need to investigate the limits (3.4) and the eigenvalues of $\mathbf{A}_N$ as $N \to \infty$. That task may be assigned to the computer for sufficiently large N. This also implies the necessary conclusions as to the exactness of the approximation of c_j by c_j^* ($j = 1, \dots, N$) and of f by f^*.

Even if conditions 3–5 of the theorem are violated from some N' on, the difference between the exact and approximate coefficients c_j ($j = 1, \dots, N$) and between f and f^*, considered for $N < N'$, will not exceed κ and $N\kappa^2 + \epsilon$, respectively.

Thus, as the order of approximation N is increased (when the conditions of the theorem are satisfied), our method may be used to obtain the Fourier coefficients of the one-dimensional density, with an error that tends to zero as $N \to \infty$. Under these conditions f^*/μ_1 will converge in the $L_2(X, \mu)$-metric to the exact solution f/μ_1. It follows, finally, that $\int_\Delta f^* d\mu_0 \to \int_\Delta f d\mu_0$ for any domain Δ, uniformly in Δ.

The problem of determining a function given the approximate values of its Fourier coefficients is ill-posed (see, for example, [9]). The convergence (whether pointwise or uniform) may be improved by using Tikhonov's method of regularization.

The software implementation of the method is relatively easy, thanks to the linearity of system (2.4). As a result, the computation of the coefficients c_j of the steady-state solution reduces to determining the inverse of the matrix $A_N^{-1}$and multiplying it by the (column) vector of free terms of system (2.4). As to the evaluation of the $c_j(t)$s as functions of time, this can be done either by solving Eqs (2.4) by computer or, relying on available explicit formulae for solving systems of linear ordinary differential equations, setting up an analytical expression for $\exp(-A_N t)$ and then doing the same for the quasi-moments $c_j^*(t)$ and density $f^*(\mathbf{x}, t)$. If $\mathbf{x}$ is a multi-dimensional vector, one can use the known $c_j^*(t)$ to obtain an approximate expression for the density of some of its components. As the formulae defining $f^*(\mathbf{x}, t)$ are rather cumbersome, it is more convenient to employ standard graphics software in order to plot the results in terms of the input data. To that end, one must first create an array of values of the density f^* for a corresponding array of $\mathbf{x}$ values.

4. EXAMPLE

Let us consider the motion of a two-dimensional mathematical pendulum with a smooth potential $\Pi(\varphi)$ depending on its position $\varphi(\mathrm{mod}\, 2\pi)$, on the assumption that the pendulum is under the action of a moment due to viscous friction and a certain random moment. This is the problem that must be solved, for example, if one is considering an unbalanced gyroscope in a Cardan suspension taking into account the moment due to viscous friction and a random moment, both moments acting along the internal axis of the Cardan suspension [10]. We shall pay particular attention to the motion of the pendulum in the uniform field of gravity.

The phase space of the system is a manifold $X = R \times S$, the local coordinates on X are the angle φ and the momentum p. We shall assume that the random moment can be written in the form $\mathbf{b}(\varphi)\mathbf{V}(t)$, where $\mathbf{V}(t)$ is a vector of steady white noise with intensity matrix ν, $\mathbf{b}(\varphi)$ is a matrix, and $\sigma(\varphi) = \mathbf{b}\nu\mathbf{b}^{\mathrm{T}}$ ($\dim \sigma = 1$) is a smooth function of φ. Then the equations of motion in dimensionless variables are as follows (ϵ denotes the coefficient of friction)

$$\varphi^{\cdot} = p, \quad p^{\cdot} = -\partial \Pi / \partial \varphi - \epsilon p + b(\varphi) V(t) \tag{4.1}$$

These equations define a diffusion process on X, controlled by the operator

$$G = p \frac{\partial}{\partial \varphi} - \left(\frac{\partial \Pi}{\partial \varphi} + \epsilon p \right) \frac{\partial}{\partial p} + \frac{1}{2} \sigma(\varphi) \frac{\partial^2}{\partial p^2}$$

We shall analyse this process using the above algorithm. We first define $L_2(X, \mu)$ in this case. Since X is the direct product of the circle S and the real line R, we can use the theorem about orthogonal systems in products [7]. As a measure μ_φ on S we take the "normalized" Lebesgue measure $d\mu_\varphi = d\varphi/(2\pi)$, and as an orthonormal basis the functions 1, $\sqrt{2}\cos\varphi$, $\sqrt{2}\sin\varphi$, $\sqrt{2}\cos 2\varphi$, $\sqrt{2}\sin 2\varphi, \ldots$

The measure on R will be the finite measure μ_p such that

$$d\mu_p = \exp[-p^2/(2\gamma)]\, dp/\sqrt{2\pi\gamma}$$

where $\gamma > 0$ is a parameter, chosen from considerations of convenience: the orthonormal basis will be the sequence of Hermite polynomials

$$H_0(p) = 1, \quad H_1(p) = p/\sqrt{\gamma}, \quad H_2(p) = (p^2 - \gamma)/(\sqrt{2}\,\gamma), \ldots$$
$$H_n(p) = (-1)^n \sqrt{\gamma^n} \exp(p^2/(2\gamma))[d^n/dp^n \exp(-p^2/(2\gamma))]/\sqrt{n!}$$

Then, by the above-mentioned theorem, the system of functions $H_m(p)\sqrt{2}\sin n\varphi$, $H_m(p)\sqrt{2}\cos n\varphi$ (where m, n are non-negative integers) is a complete orthonormal system in $L_2(X, \mu)$, $\mu = \mu_\varphi \oplus \mu_p$.

The one-dimensional density $f(\varphi, p, t)$ of the stochastic process $\varphi(t)$, $p(t)$ admits of the expansion

$$f \approx \mu_1(p) \left\{ \sum_{m=0}^{\infty} H_m(p)\left[d_m + \sqrt{2} \sum_{n=1}^{\infty} (a_{mn} \sin n\varphi + b_{mn} \cos n\varphi)\right] \right\} \tag{4.2}$$
$$\mu_1 = (2\pi)^{-3/2} \gamma^{-1/2} \exp(-p^2/(2\gamma)), \quad d_0 = 1$$

The coefficients of this expansion are given by

$$a_{mn} = MH_m(p)\sqrt{2}\sin n\varphi, \quad b_{mn} = MH_m(p)\sqrt{2}\cos n\varphi, \quad d_m = MH_m(p)$$

Confining our attention in (4.2) to index values $0 \leqslant m \leqslant N$, $0 \leqslant n \leqslant N$ (rectangular summation) and solving the resulting system of ordinary differential equations (2.4) [of order $N(2N+3)$], say by computerized methods, we obtain an approximate expression for the one-dimensional density $f(\varphi, p, t)$. As a byproduct of this procedure we obtain the coefficients a_{mn}, b_{mn}, d_m (quasi-moments) as functions of time, and then all the moments $c_{mn} = Mp^m e^{in\varphi}$ of orders up to and including N.

Let us verify the assumptions of the theorem. Since the right-hand sides of Eqs (4.1) are polynomials in p and the specific measure μ_p has been chosen, the condition $Ge_j = GH_m(p)\sqrt{2}\sin n\varphi$ (or $GH_m(p)\sqrt{2}\cos n\varphi) \in L_2(X, \mu)$ is satisfied, as Ge_j is a finite linear combination of Hermite polynomials. A direct check shows that $\chi(N) \sim N^3$. In Sec. 5 we shall show that condition (3.5) is satisfied (for an arbitrary long but finite interval of time) if $\kappa_2(N)$ increases at most as rapidly as a power of N and $\Pi(\varphi)$ and $\sigma(\varphi)$ are sufficiently smooth functions of φ.

Continuing, we confine ourselves to the motion of a pendulum in a uniform gravitational field under conditions of translational vibration in the plane of the pendulum. Let the vector of vibro-accelerations be a vector of normal steady white noise $\mathbf{V} = [V_1, V_2]'$ with intensity matrix $\boldsymbol{\nu} = \|\nu_{ij}\|$ $(i, j = 1, 2)$. Then Π and σ are analytic function of φ. Equations (4.1) will be

$$\varphi^{\cdot} = p, \quad p^{\cdot} = -\sin\varphi - \epsilon p + V_1 \sin\varphi - V_2 \cos\varphi$$

The infinite (denumerable) system of ordinary differential equations (2.3) for the unknown expansion coefficients a_{mn}, b_{mn}, d_m is

$$a^{\cdot}_{mn} = -m\epsilon a_{mn} + \frac{1}{2}\sqrt{\frac{m}{\gamma}}\,(-b_{m-1,n-1} + 2n\gamma b_{m-1,n} + b_{m-1,n+1}) + n\sqrt{(m+1)\gamma}\, b_{m+1,n} +$$
$$+ \frac{\sqrt{m(m-1)}}{4\gamma}\,[\nu_{12}(b_{m-2,n+2} - b_{m-2,n-2}) + \nu\ (a_{m-2,n+2} + a_{m-2,n-2})] \tag{4.3}$$
$$b^{\cdot}_{mn} = -m\epsilon b_{mn} - \frac{1}{2}\sqrt{\frac{m}{\gamma}}\,(-a_{m-1,n-1} + 2n\gamma a_{m-1,n} + a_{m-1,n+1}) - n\sqrt{(m+1)\gamma}\, a_{m+1,n} +$$

$$+\frac{\sqrt{m(m-1)}}{4\gamma}\,[\nu_{12}(-a_{m-2,\,n+2}+a_{m-2,\,n-2})+\nu^{-1}(b_{m-2,\,n+2}+b_{m-2,\,n-2})]$$

$$\dot{d}_m=-m\epsilon d_m-\sqrt{\frac{m}{2\gamma}}\,a_{m-1,\,1}+\frac{\sqrt{m(m-1)}}{2\sqrt{2\gamma}}\,(-\nu_{12}a_{m-2,2}+\nu^{-}b_{m-2,2})$$

$$\nu^{-}=(\nu_{22}-\nu_{11})/2,\quad \nu^{+}=(\nu_{22}+\nu_{11})/2,\quad \gamma=\nu^{+}/(2\epsilon)$$

Confining our treatment of system (4.3) to m and n such that $0\leqslant m,\ n\leqslant N$ and solving the resulting finite system of linear differential equations, we obtain an approximate expression for the density and approximate values for the quasi-moments.

One should note that Khas'minskii's theorem implies the existence of a limiting steady-state solution, i.e. the infinite system of exact equations (4.3) has an asymptotically stable point. Therefore all the coefficients a_{mn}, b_{mn}, d_m will ultimately tend to definite constant values, which determine the density f of the steady-state solution. For any finite closure, the trace of the matrix of the linear system thus obtained will be negative (it is equal to $-\epsilon N(N+1)(2N+1)/2$). But this means that the phase flow of the equations reduces volumes.

Numerical experiments on a personal computer, using the scientific software package NALIB (developed at the Institute of Informatics Problems of the Academy of Science of the U.S.S.R.) have shown that when $N=2,\ldots,6$ (for definite values of ϵ, γ, ν) all the real parts of the eigenvalues of $\mathbf{A}_N$ are negative (all the eigenvalues are simple) and $\kappa_2(N)$ depends only very slightly on N. All computations were repeated for two other measures μ_p with densities 1 and $\exp(p^2/(2\zeta))$, respectively (the orthogonal basis on the real line in both cases consisted of the Hermite functions), guaranteeing the validity of the conditions $f/\mu_1\in L_2(X,\mu)$ and $\partial^{k+r}(f/\mu_1)/\partial\varphi^k\partial p^r\in L_2(X,\mu)$. Since the results were the same, these cases are not discussed here.

Figure 1 shows the form of the one-dimensional density f_{st} of the steady-state distribution in the cube $\{|\varphi|\leqslant 5,\ |p|\leqslant 10,\ 0\leqslant z\leqslant 0.5\}$, in the case $\epsilon=\gamma=1$, $N=4$, and $\nu_{11}=3$, $\nu_{22}=1$, $\nu_{12}=0.1$. An interesting consequence of the fact that $\nu_{11}\neq\nu_{22}$ and $\nu_{12}\neq 0$ is that the distribution is multimodal. The modes correspond to the points $\varphi=m\pi$ $(m=0,\pm 1,\ldots)$. Thus, owing to the inhomogeneity of the vibration, there is a higher probability for the pendulum to be in its vertical position $\varphi=\pi$. For comparison, Fig. 2 shows the form of the one-dimensional steady-state solution in the case $\epsilon=\gamma=1$, $\nu_{11}=\nu_{22}=2$ [see formula (4.4)]. For any p, the distribution on the circle $S=\{\varphi\}$ is in this case unimodal ($\varphi=0$ is the mode, $\varphi=\pi$ the antimode [11]).

Figure 3 shows the probability density of the angle φ in the case $\epsilon=\gamma=1$, $N=4$, $\nu_{11}=\nu_{22}=2$). The solid curve represents the steady-state solution, and the dashed curve a transient at the time $t=½$. The initial density was taken to be the function $f_0(\varphi)=[1+\sin(\varphi+\pi/3)]/(2\pi)$.

Table 1 lists the onset time T of steady behaviour, for a few values of the parameters of the problem.
The initial density was taken to be $f_0(\varphi,p)=f_0(\varphi)\exp(-p^2/2)/\sqrt{2\pi}$.

Some information may be derived by considering only the "first" approximation, when the only coefficients

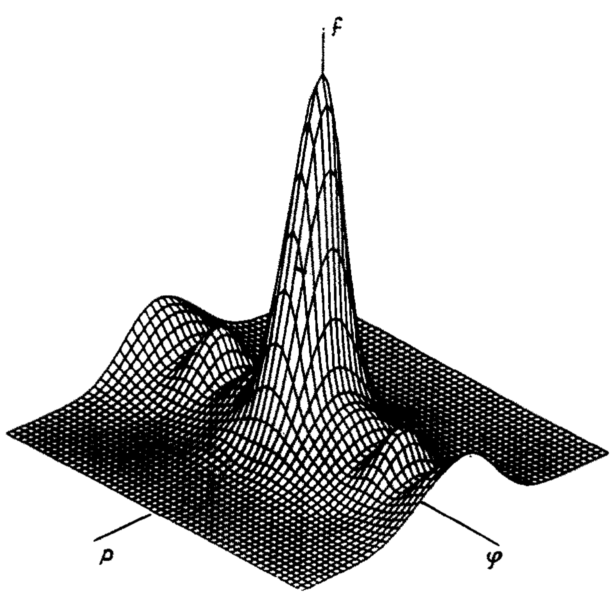

FIG. 1.

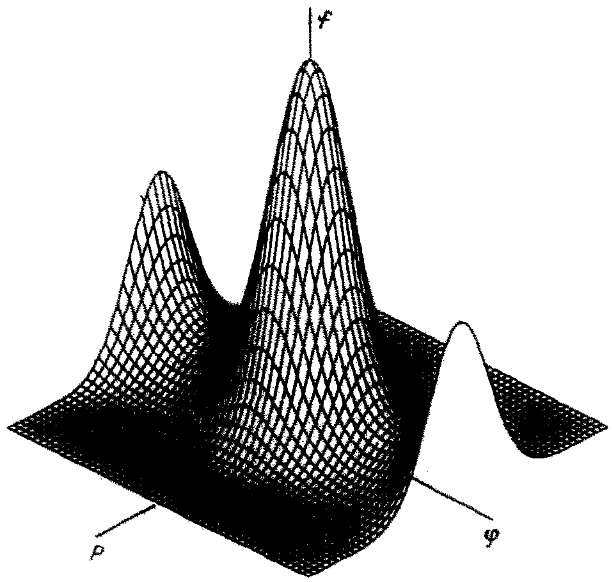

Fig. 2.

taken into account from (4.2) are a_{01}, b_{11}, b_{01}, a_{11}, d_1. Then system (2.4) includes five equations, which split into three independent subsystems: for a_{01}, b_{11}, for b_{01}, a_{11} and for d_1. The characteristic values of the first two subsystems are identical: $\lambda_{1,2} = -\epsilon/2 \pm \sqrt{\epsilon^2/4 - \gamma}$, while the characteristic value of the third is $-\epsilon$. Hence, in particular, it is evident that the onset of steady behaviour will occur earlier if $\epsilon^2/4 - \gamma < 0$, i.e. $\nu > \epsilon^3/2$. This is in good agreement with the data of numerical experiments (see above).

One more property of system (4.3) is worth noticing. If $\nu_{12} = 0$, it splits into two independent subsystems. The first includes a_{mn} for even m and b_{mn} for odd m. The second includes a_{mn} for odd m and b_{mn} for even m.

We notice that when the steady-state density f_{st} is known, our method yields a representation of the steady-state values of the quasi-moments $c_j = \int_X e_j f_{st} d\mu_0$ as solutions of an infinite system of linear algebraic equations [one should set $\dot{c}_j = 0$ in (2.2)]. Sometimes this solution may be found in the form of continued fractions. In such cases one obtains exact, rapidly convergent formulae for the integrals $\int_X e_j f_{st} d\mu_0$, though it may be difficult to evaluate these integrals directly.

To illustrate this, let us consider the example of a pendulum in a homogeneous gravitational field, when $\nu_{12} = 0$, $\nu_{11} = \nu_{22} = \nu$. In this case, the exact expressions for the density of the steady process $\varphi(t)$, $p(t)$ (in the narrow sense) is well known

$$f_{st}(\varphi, p) = \theta \exp[-\gamma^{-1}(p^2/2 - \cos\varphi)], \quad \theta = \text{const}, \quad \gamma = \nu/(2\epsilon) \tag{4.4}$$

It follows from this formula that φ and p are statistically independent in the limit. The momentum p is normally distributed with zero mean and variance γ, while the probability density of the angle φ (the density of the measure on the unit circle) is given by $f_1(\varphi) = \theta_1 \exp(\gamma^{-1}\cos\varphi)$, $\theta_1 = \text{const}$.

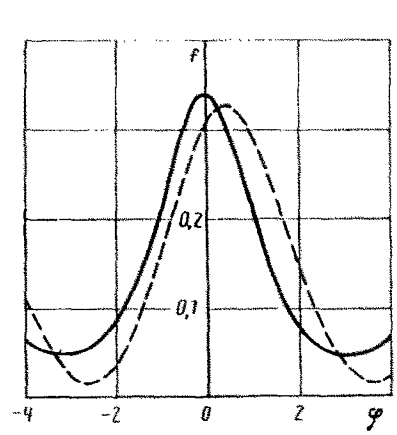

Fig. 3.

TABLE 1

ϵ	1	1	1	2	4	0.25	0.5	1	2
ν_{11}	2	4	8	2	2	2	3	3	3
ν_{12}	0	0	0	0	0	0	0.1	0.1	0.1
ν_{22}	2	4	8	2	2	2	1	1	1
γ	1	2	4	0.5	0.25	16	2	1	0.5
T	11.5	12.5	13	28.5	35	42	26.5	15	30

The numbers

$$c_n = \int_{-\pi}^{\pi} e^{in\varphi} f_1(\varphi)d\varphi = b_{0n} + ia_{0n} \qquad (n = 0, \pm 1, \pm 2, \dots)$$

are known as the power moments of the measure $\mu(d\mu = f_1(\varphi)d\varphi)$ or trigonometric moments. The sequence c_n is also called the characteristic function of φ [11].

The steady-state values of the coefficients a_{mn}, b_{mn}, d_m [the equilibrium position of system (4.3)] may be determined. The only non-zero coefficients are b_{0n}, for which we obtain the following system of linear algebraic equations

$$-\sqrt{2} + 2\gamma b_{01} + b_{02} = 0, \quad \dots, \quad -b_{0,n-1} + 2n\gamma b_{0,n} + b_{0,n+1} = 0, \dots \tag{4.5}$$

Solutions of these equations are found as continued fractions, yielding expressions for the steady-state values of the trigonometric moments

$$M\cos\varphi = \int_0^{2\pi} f_1(\varphi)\cos\varphi d\varphi = b_{01}/\sqrt{2} = 1/2\gamma + 1/4\gamma + 1/6\gamma + \dots$$

$$M\cos 2\varphi = \int_0^{2\pi} f_1(\varphi)\cos 2\varphi d\varphi = b_{02}/\sqrt{2} = 1/1 + 2(2\gamma)^2 + 2\gamma/6\gamma + 1/8\gamma + 1/10\gamma + \dots$$

Although the approximate results described above are guaranteed to be close to the exact solution over a finite time interval (which will be longer, the greater N), the case of $\nu_{12} = 0$, $\nu_{11} = \nu_{22} = \nu$ demonstrates that the approximation may remain good over an infinite time interval. Here the convergence of the approximate value of the steady-state f_{st}^* to its exact counterpart f_{st} as N increases is as rapid as for the continued fractions b_{01} and b_{02}.

5. CONCLUSION

We will now consider the function $F(\varphi, p, t) \equiv f(\varphi, p, t)/(\sqrt{2}\mu_1(p)) \in L_2(X, \mu)$ and its Fourier series

$$F(\varphi, p, t) \sim \sum_{m=0}^{\infty} H_m(p)[d_m + \sum_{n=1}^{\infty} (a_{mn}\sin n\varphi + b_{mn}\cos n\varphi)] \quad (d_0 = 1) \tag{5.1}$$

Let us estimate the remainder $\epsilon(N, t)$ of the Parseval series for F. Let F have k continuous derivatives with respect to φ, and r continuous derivatives with respect to p. Differentiating the series (5.1) successively k times with respect to φ, and r times with respect to p, and comparing the coefficients of the series thus obtained

$$\frac{\partial^{k+r}F}{\partial\varphi^k \partial p^r} \sim \sum_{m=0}^{\infty} H_m(p)[d_m^* + \sum_{n=1}^{\infty} (a_{mn}^*\sin n\varphi + b_{mn}^*\cos n\varphi)]$$

with those of the original series, we see that, depending on the parity of k, one has either

$$a_{mn}^* = \pm n^k a_{m+r,n}R, \quad b_{mn}^* = \pm n^k b_{m+r,n}R, \quad d_m^* = d_{m+r}R$$
$$R = 2^r(m+r)(m+r-1)\dots(m+1)$$

or

$$a^*_{mn} = \pm n^k b_{m+r,\,n} R, \quad b^*_{mn} = \pm n^k a_{m+r,\,n} R$$

Let us assume that $\partial^{k+r} F/\partial\varphi^k \partial p^r \in L_2(X, \mu)$ for any t. Then, using the Cauchy–Schwarz inequality (as in [12]) for series, we obtain

$$\epsilon(N, t) = \sum_{m=0}^{N} \sum_{n=N+1}^{\infty} h_{mn} + \sum_{m=N+1}^{\infty} (d_m^2 + \sum_{n=1}^{\infty} h_{mn}) =$$

$$= 2^{-2r} [\sum_{m=r}^{N} \sum_{n=N+1}^{\infty} n^{-2k} M_m g_{mn} + \sum_{m=N+1}^{\infty} M_m (d^{*2}_{m-r} + \sum_{n=1}^{\infty} n^{-2k} g_{mn})] \leqslant$$

$$\leqslant 2^{-2r} [(\sum_{m=r}^{N} M_m^2 \sum_{n=N+1}^{\infty} n^{-4k})^{1/2} (\sum_{m=r}^{N} \sum_{n=N+1}^{\infty} g_{mn}^2)^{1/2} + (\sum_{m=N+1} M_m^2)^{1/2} (\sum_{m=N+1} d^{*4}_{m-r})^{1/2} +$$

$$+ (\sum_{m=N+1}^{\infty} M_m^2 \sum_{n=1}^{\infty} n^{-4k})^{1/2} (\sum_{m=N+1}^{\infty} \sum_{n=1}^{\infty} g_{mn}^2)^{1/2}] \leqslant \rho N^{-2\min(k,\,r)+1/2} \epsilon^*(N, t), \quad \rho = \text{const}$$

$$h_{mn} = a_{mn}^2 + b_{mn}^2, \quad g_{mn} = a^{*2}_{m-r,\,n} + b^{*2}_{m-r,\,n}, \quad M_m = [m(m-1)\ldots(m-r+1)]^{-2}$$

where $\epsilon^*(N, t)$ is an estimate for the remainder term (in the rectangular summation method assumed here) of the Parseval series of $\partial^{k+r} F/\partial\varphi^k \partial p^r$.

If $\sup_t \epsilon^*(N, t) \to 0$ as $N \to \infty$, i.e. the series (5.1) of the function $\partial^{k+r} F/\partial\varphi^k \partial p^r$ is uniformly convergent, the series (5.1) for F is also uniformly convergent and the required estimate is $\epsilon(N) \sim N^{-2\min(k,r)+1/2}$. However, this condition is not easy to verify, so we limit ourselves to as long but finite a time interval $[t_0, T]$ as required. Let

$$\sup_{t \in [t_0, T]} \epsilon^*(N, t) = \eta(N).$$

Over this interval, then, we have $\epsilon(N) \leqslant \rho\eta N^{-2\min(k,r)+1/2}$ Condition (3.5) (translated into the vector indexing adopted in this section) will be

$$\lim_{N \to \infty} [N(2N+3)]^5 \kappa_2^2(N) N^{-2\min(k,\,r)+1/2} \chi(N) = 0 \tag{5.2}$$

Consequently, if $\kappa_2^2\chi$ increases as $N \to \infty$ at most as rapidly as $N^{2\min(k,r)-10-1/2}$, then condition (5.2) is satisfied.

In conclusion, we would like to point out that these results may be generalized to the case in which the vector of random forces $\mathbf{F}$ may be written as $\mathbf{F} = \mathbf{b}(\mathbf{q})\pi(t)$, where $\pi(t) \in R^{l_1}$ is a vector of random functions satisfying the shaping filter equation

$$\pi^{\cdot} = \mathbf{a}(\pi) + \mathbf{b}(\pi)\mathbf{V}(t)$$

where $\mathbf{V}(t)$ ($\dim V = 1$) is a vector of normally distributed white noise with intensity matrix ν. In that case $X = T^*Q \times R^{l_1}$, $\mathbf{x} = [\mathbf{q}^t \mathbf{p}^t \pi]$.

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